The Marriage Problem and the Fate of Bachelors

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(October 8, 1997; February 1, 2008)

In the marriage problem, a variant of the bi-parted matching problem, each member has a 'wish-list' expressing his/her preference for all possible partners; this list consists of random, positive real numbers drawn from a certain distribution. One searches the lowest cost for the society, at the risk of breaking up pairs in the course of time. Minimization of a global cost function (Hamiltonian) is performed with statistical mechanics techniques at a finite fictitious temperature.

The problem is generalized to include bachelors, needed in particular when the groups have different size, and polygamy. Exact solutions are found for the optimal solution ($T = 0$). The entropy is found to vanish quadratically in T. Also other evidence is found that the replica symmetric solution is exact, implying at most a polynomial degeneracy of the optimal solution.

Whether bachelors occur or not, depends not only on their intrinsic qualities, or lack thereof, but also on global aspects of the chance for pair formation in society.

I. INTRODUCTION

The stable marriage is a well-known optimization problem "women", and pairs are formed between them. Many other applications exist, such as the job-market (employers and employees), the housing market, etc. Each man ranks the women according to his preference, number one being his most preferred candidate; likewise, each woman ranks the men. The cost function for a configuration of marriages is the sum of the rankings of the actual partners on the two lists. In the past most studies have been concerned with algorithms for finding stable pairing solutions. Indeed, given the rules according to which marriages may be ended and started at new, a stable situation may be reached where it brings no benefit to break up existing relations for starting other ones. For a recent study of this subject, see Oméro et al [3].

Since there is intrinsically quenched disorder (the lists, see below), many such stable solutions are actually meta-stable in the sense that their cost function is not minimal. If the rules for breaking pairs are then softened, the society will go to a state of lower cost, i.e., more happiness. In some sense this is just what happens in reality. Indeed, due to changes in morality, marriage rules are weakened in the course of time. One century ago it was in most societies both morally and economically almost impossible to separate married couples, whereas nowadays it is a generally accepted phenomenon in most Western and several Eastern societies. In line of the principle of freedom of the individual, many American movies and television soap series nowadays even advocate that a person has the right to improve his/her own situation, no matter what cost this brings to the present partner, to the surroundings of the aimed new partner, or to society. To give a proper description of the total happiness of society, one should then, however, also take into account the cost for breaking up existing families. Such a more complete description would, naturally, put limits to individual freedom.

Let us consider a group A of N men and a group B of M woman. In reality the ratio $\nu = M/N$ will slightly exceed unity. The man i has a wish-list, that is to say he evaluates all the women in group B and ranks them into a preference order. He will prefer, of course, the number one on his list, but that woman may not have reciprocal intentions. Our task is to find a best global arrangement in which the total "happiness" is optimal. This calls for a cost function which we shall refer to as the Hamiltonian. In each arrangement preferably one man is paired to one woman. However, the possibility of remaining single is needed if the groups have a different number of members. Also polygamy can be permitted with assigned penalty weights.

Globally stable solutions can be only found for small systems using exact enumeration methods. It is not hard to imagine that, as the system becomes larger, many conflicting wishes make a satisfactory solution harder to find. A systematical analysis is called for, which is conveniently done in terms of a partition sum at a fictitious temperature T; finally the $T \to 0$ limit should be taken. There is a connection with spin glasses where many different but roughly equivalent states occur. It was this connection which led in the mid-eighties to the replica analysis of several combinatorial problems [4], such as the bi-parted matching problem by Orland [5], the mono-parted matching problem and the traveling salesman problem by Mézard and Parisi [6].

The wish list can be implicitly given for the man i. For a given distribution he draws a (random) set of real positive numbers ℓ^a_{ij}, that gives the energy cost for pairing with woman j. This set can in principle be arranged into

a increasing order, with the most preferred partner having the least cost. Likewise, each woman j has a different (random) list ℓ_{ij}^b giving the energy cost to form a pair with man i.

In standard studies the cost of a configuration of marriages is the sum of rankings on the lists. For matching two groups of N members each person therefore brings an integer cost C, $1 \le C \le N$, that equals the rank of his actual partner on his wish list. These integer costs are not so realistic, however. Number ten on a large list of candidates will almost be just as acceptable as number one, and not ten times less. For this reason we will follow Orland and Mézard-Parisi, who have considered non-integer costs, ℓ_{ij}.

The total cost for pair (i,j) is then $\ell_{ij} = \alpha\ell_{ij}^a + (1-\alpha)\ell_{ij}^b$. Note that in general $\ell_{ij}^a \neq \ell_{ij}^b$ since intentions are not in general reciprocal. α characterizes another asymmetry of the problem: when $\alpha = 1$ the cost function only represents men's interest, with that of women totally ignored, vice versa $\alpha = 0$ women's wish prevails. Besides these extremal men or women dominated situations, the case $\alpha = 1/2$ represents the symmetrical (ideal) balance between the two groups. In society α is slightly tilt to men's advantage.

Apart from this, if the man i remains single, this costs an energy A_i. Basically, if $A_i > \min_j \ell_{ij}^a$ then it is profitable for him to form a pair. Likewise, the cost for woman j to remain single is B_j. We shall also allow for polygamy in a grand-canonical fashion. If man i marries $k \ge 2$ women, there is an additional energy cost $(k-1)\mu_{1i}$, where μ_{1i} plays the role of a chemical potential. Similarly there is a cost $(k-1)\mu_{2j}$ when woman j marries $k \ge 2$ man. Also the μ's can be taken as random variables, or may just be fixed.

The goal of the problem is to find the best possible solution, i.e., the one that has lowest energy ("cost") for the whole society. It remains to be seen in how far this is the best solution for a typical individual, and which dynamics has the best state as stable solution.

For equally large groups ($M = N$) and and no singles ($A_i = B = \infty$) nor polygamy ($\mu_{1i} = \mu_{2j} = \infty$) this marriage problem is closely related to the bipartite matching problem studied by Orland [5] and by Mézard and Parisi [6]. The major difference lies in the asymmetry of energy costs $\ell_{ij}^a \neq \ell_{ij}^b$, the new features such like being single or polygamy presents also new questions. Our analysis will follow these approaches as much as it is possible.

The setup of the paper is as follows. In section II we describe some simple scaling aspects of the problem. In section III we describe a detailed replica analysis. In section IV we make a low T analysis. We close with a discussion and summary. In the appendix we consider the energy fluctuations.

II. SCALING ANALYSIS OF THE PROBLEM

A state is fully characterized by the set of numbers $\{n_{ij}\}$, where $n_{ij} = 1$ if the pair (i,j) is formed and zero else. The total number of partners of man i is $n_i = \sum_j n_{ij}$. Its typical value is $n_i = 1$ (man i finds a woman/a woman finds him); however, $n_i = 0$ (man i remains single) or $n_i \ge 2$ (he is a polygamist) may also occur. In a similar way $m_j \equiv \sum_i n_{ij}$ describes the fate of woman j. The interesting case is where n_i and m_j are annealed variables, so that both possibilities can be weighted thermally.

The energy of a state is

$$\mathcal{H} = \sum_{ij} n_{ij}\ell_{ij} + \sum_i A_i\delta_{n_i,0} + \sum_i \sum_{k=2}^{\infty}(k-1)\mu_{1i}\delta_{n_i,k} + \sum_j B_j\delta_{m_j,0} + \sum_j \sum_{k=2}^{\infty}(k-1)\mu_{2j}\delta_{m_j,k} \tag{1}$$

Let us first consider the scaling for the pairs formed. The cost for pair (i,j) is

$$\ell_{ij} = \alpha\ell_{ij}^{(a)} + (1-\alpha)\ell_{ij}^{(b)} \tag{2}$$

In order to form a pair, this combined value should be as low as possible. First consider $\alpha = 1$. Then typically the lowest $\ell_{ij}^{(a)}$ is $\ell_{min}^{(a)}$ is given by

$$\int_0^{\ell_{min}^{(a)}} p_a(\ell)d\ell \approx \frac{1}{N} \tag{3}$$

Since we shall take

$$p_a(\ell) = \frac{\ell_a^r e^{-\ell}}{\Gamma(r_a+1)} \tag{4}$$

this implies $\ell_{min}^{(a)} = [\Gamma(r_a+1)/N]^{1/(r_a+1)}$. The pairing energy would be equal

$$U_{pairs} = N\ell^{(a)}_{min} \sim N^{r_a/(r_a+1)} \tag{5}$$

Typically, a man would find a woman that is high on the top of his list. The women would accept this, since their wishes have no effect what so ever.

Now, if woman's preference is also taken into account ($\alpha < 1$) things change. For $\alpha = 1/2$ the distribution for the total ℓ is:

$$p(\ell) = \frac{2^{r+1}\ell^r e^{-2\ell}}{\Gamma(r+1)} \qquad r = r_a + r_b + 1 \tag{6}$$

while for general α the small-ℓ behavior is

$$p(\ell) \approx \frac{\ell^r}{\alpha^{r_a+1}(1-\alpha)^{r_b+1}\Gamma(r+1)} \qquad (\ell \to 0) \tag{7}$$

The typical value $\ell_{typ} \sim \ell_{min} \sim N^{-1/(r+1)}$ now implies that the typical partner is located on position k of the wish list related as $\int_0^{\ell_{typ}} d\ell p_a(\ell) = k/N$, yielding $k \sim N^{(r_b+1)/(r+1)}$. For the symmetric case $r_a = r_b = (r-1)/2$ this becomes $k \sim \sqrt{N}$. So for the two-side weighted case the wishes of each individual are quite moderately fulfilled: the ideal partner is not within reach. She/he has typically has different wishes, which excludes formation of the pair. For the bulk of pairs the partners are only moderately biased towards each other. This is the price society has to pay for individual freedom.

The ground state energy will scale as $U_0 \sim N\ell_{typ} \sim N^{r/(r+1)}$. This implies that bachelors and polygamist s only play an interesting role when their costs ($A_i, B_j, \mu_{1,i}$ and $\mu_{2,j}$) scale as $U_0/N \sim N^{-1/(r+1)}$. If the costs have a larger magnitude, these options do not occur provided the groups have equal size; if they are smaller, no pairs will be formed. Notice that these scalings are not universal to the individual, but depend on the exponent r entering the distribution (7) of costs for pairings in the whole societies.

III. THE PARTITION SUM AND REPLICA'S

We consider the problem in a statistical mechanics approach. We thus consider the sum over all configurations of Boltzmann factors $\exp(-\beta E)$ at given temperature $T = 1/\beta$. The optimal solution has lowest energy; it will be dominant in the limit $T \to 0$. The statistical approach also yields the entropy, related to the degeneracy of the optimal solution, as well as the average numbers of married man etc.

To simplify notation we first take sure values for A_i, B_j, $\mu_{1\,i}$, $\mu_{2\,j}$. The partition sum can be written as

$$\begin{aligned} Z &= \prod_{i=1}^{N}\prod_{j=1}^{M}\sum_{n_{ij}=0}^{1}\sum_{n_i,m_j=0}^{\infty} e^{-\beta H} \\ &= \sum_{n_{ij}=0}^{1}\sum_{n_i,m_j=0}^{\infty} e^{-\beta\sum_{i=1}^{N}\{(A+\mu_1)\delta_{n_i,0}+(n_i-1)\mu_1\}} e^{-\beta\sum_{j=1}^{M}\{(B+\mu_2)\delta_{m_j,0}+(m_j-1)\mu_2\}} \\ & e^{\,-\beta\sum_{ij} n_{ij}\ell_{ij}} \prod_{i=1}^{N}\delta(n_i - \sum_{j=1}^{M} n_{ij}) \prod_{j=1}^{M}\delta(m_j - \sum_{i=1}^{N} n_{ij}) \end{aligned} \tag{8}$$

where the δ's are Kronecker δ-functions. Repeatedly using the integral representation

$$\delta(a-b) = \int_0^{2\pi} \frac{\mathrm{d}\lambda}{2\pi} e^{i\lambda(a-b)} \tag{9}$$

one arrives at a form where the sums can be carried out. After doing that one gets

$$Z = \prod_{i=1}^{N}\int_0^{2\pi}\frac{d\lambda_i}{2\pi}\frac{e^{-\beta A}+(1-e^{-\beta(A+\mu_1)})e^{i\lambda_i}}{1-e^{-\beta\mu_1+i\lambda_i}} \prod_{j=1}^{M}\int_0^{2\pi}\frac{d\mu_j}{2\pi}\frac{e^{-\beta B}+(1-e^{-\beta(B+\mu_2)})e^{i\mu_j}}{1-e^{-\beta\mu_2+i\mu_i}} \prod_{i,j=1}^{N,M}(1+e^{-i\lambda_i-i\mu_j-\beta\ell_{ij}}) \tag{10}$$

In order to calculate the quenched average free energy one employs the replica trick:

$$\overline{F} = -T\overline{\ln Z} = -T \lim_{n\to 0} \frac{\overline{Z^n} - 1}{n} \tag{11}$$

One thus replicates the partition sum n times and uses for each pair (i, j)

$$\prod_{\alpha=1}^{n} (1 + e^{-i\lambda_i^\alpha - i\mu_j^\alpha - \beta\ell_{ij}}) = 1 + \sum_{p=1}^{n} \sum_{1\le\alpha_1<\cdots<\alpha_p\le n} e^{-i(\lambda_i^{\alpha_1}+\cdots+\lambda_i^{\alpha_p}) - i(\mu_j^{\alpha_1}+\cdots+\mu_j^{\alpha_p}) - p\beta\ell_{ij}} \tag{12}$$

Now the quenched average over ℓ_{ij} can be carried out. One gets

$$\overline{e^{-p\beta\ell_{ij}}} = \int d\ell_{ij}^a p(\ell_{ij}^a) d\ell_{ij}^b p(\ell_{ij}^b) e^{-p\beta(\alpha\ell_{ij}^a + (1-\alpha)\ell_{ij}^b)} \equiv \frac{g_p}{N} \tag{13}$$

This defines

$$g_p = \frac{N}{(1+\alpha p\beta)^{r_a+1}(1+(1-\alpha)p\beta)^{r_b+1}} \approx \frac{N}{(p\beta)^{1+r}\alpha^{r_a+1}(1-\alpha)^{r_b+1}} \tag{14}$$

where $r = r_a + r_b + 1$. In terms of r our g_p has a similar form as in the matching problem [5] [6].

We shall confine ourselves to low temperatures, where we scale temperature with N such that

$$\hat{T} = T\left(\frac{N}{\alpha^{r_a+1}(1-\alpha)^{r_b+1}}\right)^{1/(r+1)} \tag{15}$$

is of order unity. This implies that

$$g_p \approx \left(\frac{\hat{T}}{p}\right)^{r+1} \tag{16}$$

is also of order unity. We then get to leading order in N

$$\overline{Z^n} = \prod_{i,\alpha} \int \frac{d\lambda_i^\alpha}{2\pi} \frac{e^{-\beta A} + (1 - e^{-\beta(A+\mu_1)})e^{i\lambda_i^\alpha}}{1 - e^{-\beta\mu_1 + i\lambda_i^\alpha}} \prod_{j,\alpha} \int \frac{d\mu_j^\alpha}{2\pi} \frac{e^{-\beta B} + (1 - e^{-\beta(B+\mu_2)})e^{i\mu_j^\alpha}}{1 - e^{-\beta\mu_2 + i\mu_j^\alpha}} \; e^S \tag{17}$$

with

$$S = \frac{1}{N} \sum_{ij} \sum_{p=1}^{n} g_p \sum_{1\le\alpha_1<\cdots<\alpha_p\le n} e^{-i(\lambda_i^{\alpha_1}+\cdots+\lambda_i^{\alpha_p}) - i(\mu_j^{\alpha_1}+\cdots+\mu_j^{\alpha_p})} \tag{18}$$

The i, j-dependence of $\exp S$ can be decoupled by a Hubbard-Stratonovich-type transformation

$$\begin{aligned} e^S = & \prod_{p=1}^{n} \prod_{\{\alpha_r\}} \int \mathrm{d}P_{\alpha_1\cdots\alpha_p} \mathrm{d}Q_{\alpha_1\cdots\alpha_p} \exp\left[-N \sum_{p=1}^{n} \frac{1}{g_p} \sum_{1\le\alpha_1<\cdots<\alpha_p\le n} Q_{\alpha_1\cdots\alpha_p} P_{\alpha_1\cdots\alpha_p}\right] \\ & \times \exp \sum_{p=1}^{n} \sum_{1\le\alpha_1<\cdots<\alpha_p\le n} \{P_{\alpha_1\cdots\alpha_p} \sum_{i=1}^{N} e^{-i(\lambda_i^{\alpha_1}+\cdots+\lambda_i^{\alpha_p})} + Q_{\alpha_1\cdots\alpha_p} \sum_{j=1}^{M} e^{-i(\mu_j^{\alpha_1}+\cdots+\mu_j^{\alpha_p})}\} \end{aligned} \tag{19}$$

Now all i, j indices are decoupled. The λ_i integrals all yield the same factor. The replicated partition sum thus becomes equal to a P, Q integral of a function $\exp[-N\Phi(P, Q)]$. Because N is large, we may approximate the integral by its saddle point value. Therefore the replicated free energy $F_n = -T \ln \overline{Z^n}$ is just equal to this saddle point value. We thus obtain

$$\frac{\beta F_n}{N} = \sum_{p=1}^{n} \frac{1}{g_p} \sum_{1\le\alpha_1<\cdots<\alpha_p\le n} Q_{\alpha_1\cdots\alpha_p} P_{\alpha_1\cdots\alpha_p} - z_n^a - \nu z_n^b \tag{20}$$

where the latter two objects are defined as

$$e^{z_n^a}=\prod_{\alpha=1}^{n}\left[\int_0^{2\pi}\frac{d\lambda_\alpha}{2\pi}\frac{e^{-\beta A}+(1-e^{-\beta(A+\mu_1)})e^{i\lambda_\alpha}}{1-e^{-\beta\mu_1+i\lambda_\alpha}}\right]\exp\sum_{p=1}^{n}\sum_{1\le\alpha_1<\cdots<\alpha_p\le n}P_{\alpha_1\cdots\alpha_p}e^{-i(\lambda_{\alpha_1}+\cdots+\lambda_{\alpha_p})} \tag{21}$$

and similarly

$$e^{z_n^b}=\prod_{\alpha=1}^{n}\left[\int_0^{2\pi}\frac{d\mu_\alpha}{2\pi}\frac{e^{-\beta B}+(1-e^{-\beta(B+\mu_2)})e^{i\mu_\alpha}}{1-e^{-\beta\mu_2+i\mu_\alpha}}\right]\exp\sum_{p=1}^{n}\sum_{1\le\alpha_1<\cdots<\alpha_p\le n}Q_{\alpha_1\cdots\alpha_p}e^{-i(\mu_{\alpha_1}+\cdots+\mu_{\alpha_p})} \tag{22}$$

Following Orland and Mézard-Parisi we assume that the relevant saddle point has replica symmetry, viz. $Q_{\alpha_1\cdots\alpha_p}=Q_p$, $P_{\alpha_1\cdots\alpha_p}=P_p$. Then the λ_α integrals can be replaced by contour integrals over $z_\alpha=\exp(-i\lambda_\alpha)$, and the integral can be evaluated from the poles at $z_\alpha=0$ and $\exp(-\beta\mu_1)$. This yields

$$\begin{aligned}
e^{z_n^a}&=e^{n\beta\mu_1}\prod_{\alpha=1}^{n}\sum_{z_\alpha=0,e^{-\beta\mu_1}}\left[(-1+e^{-\beta A-\beta\mu_1})\delta_{z_\alpha,0}+\delta_{z_\alpha,e^{-\beta\mu_1}}\right]\exp\sum_{p=1}^{n}P_p\sum_{1\le\alpha_1<\cdots<\alpha_p\le n}z_{\alpha_1}\cdots z_{\alpha_p}\\
&=e^{n\beta\mu_1}\sum_{k=0}^{n}\binom{n}{k}(-1+e^{-\beta A-\beta\mu_1})^k\exp\sum_{p=1}^{n-k}P_p\binom{n-k}{p}e^{-\beta\mu_1 p}\\
&=e^{n\beta\mu_1}\sum_{k=0}^{\infty}\binom{n}{k}(-1)^k(1-e^{-\beta A-\beta\mu_1})^k\exp\sum_{p=1}^{\infty}(-1)^pP_p\frac{\Gamma(k+p-n-i0)}{\Gamma(k-n-i0)p!}e^{-\beta\mu_1 p}
\end{aligned} \tag{23}$$

where $i0=i\epsilon$ with $\epsilon\to0$ first. In the last step extension of the sums to ∞ was allowed because the extra terms are identically zero. So far n has been integer. Now we can continue this expression for $n\to0$. There are two types of contributions: the $k=0$ term equals $1+\mathcal{O}(n)$, while the $k\ge1$ are all $\mathcal{O}(n)$. Indeed, for those terms one uses

$$\binom{n}{k}=\frac{\Gamma(n+i0+1)}{\Gamma(n-k+1)k!}=\frac{\Gamma(n+1)\sin(\pi(k-n-i0))\Gamma(k-n-i0)}{\pi k!}\to\frac{n(-1)^{k-1}}{k}(1-n\psi(k)+n\psi(1)+\cdots) \tag{24}$$

In the limit $n\to0$ one thus gets

$$z_n^a=nz_a+n^2z_a^{(2)}+\mathcal{O}(n^3) \tag{25}$$

The result for z_a is presented in eq. (28), while $z_a^{(2)}$ is presented in eq. (78)

Likewise $z_b\equiv\lim_{n\to0}z_n^b/n$ is given in eq. (29).

Further we find, using eq. (24),

$$\sum_{p=1}^{n}\frac{1}{g_p}\sum_{1\le\alpha_1<\cdots<\alpha_p\le n}Q_{\alpha_1\cdots\alpha_p}P_{\alpha_1\cdots\alpha_p}=\sum_{p=1}^{\infty}\frac{Q_pP_p}{g_p}\binom{n}{p}\to n\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p}\frac{Q_pP_p}{g_p} \tag{26}$$

A. The quenched free energy

Now the limit $n\to0$ can be taken simply. The quenched average free energy $F=\lim_{n\to0}F_n/n$ thus follows from eq. (20) as

$$\frac{\beta F}{N}=\sum_{p=1}^{\infty}\frac{Q_pP_p}{g_p}\frac{(-1)^{p-1}}{p}-z_a(A,\mu_1)-\nu z_b(B,\mu_2) \tag{27}$$

with

$$z_a(A,\mu_1)=\beta\mu_1+\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p}P_pe^{-p\beta\mu_1}-\sum_{k=1}^{\infty}\frac{(1-e^{-\beta(A+\mu_1)})^k}{k}\exp\left\{\sum_{p=1}^{\infty}P_p\frac{(-1)^p\Gamma(k+p)}{\Gamma(k)p!}e^{-p\beta\mu_1}\right\} \tag{28}$$

and

$$z_b(B,\mu_2) = \beta\mu_2 + \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p} Q_p e^{-p\beta\mu_2} - \sum_{k=1}^{\infty} \frac{(1-e^{-\beta(B+\mu_2)})^k}{k} \exp\left\{\sum_{p=1}^{\infty}(-1)^p Q_p \frac{\Gamma(k+p)}{\Gamma(k)p!} e^{-p\beta\mu_2}\right\} \tag{29}$$

In section 2 we have discussed the possibility that A_i, B_j, $\mu_{1\,i}$ and $\mu_{2\,j}$ are independent random variables. For that case the analysis goes along the same steps; one only has to average the present expressions for z_a and z_b with respect to these variables.

In the case where these variables are non-random, and only the pair costs $\ell_{ij}^{(a,b)}$ are random, the saddle point equations read:

$$Q_p = g_p e^{-p\beta\mu_1} + p g_p e^{-p\beta\mu_1} \sum_{k=1}^{\infty} \frac{[1-e^{-\beta(A+\mu_1)}]^k}{k} \frac{\Gamma(k+p)}{\Gamma(k)p!} e^{-P(k)} \tag{30}$$

and

$$P_p = \nu g_p e^{-p\beta\mu_2} + \nu p g_p e^{-p\beta\mu_2} \sum_{k=1}^{\infty} \frac{[1-e^{-\beta(B+\mu_2)}]^k}{k} \frac{\Gamma(k+p)}{\Gamma(k)p!} e^{-Q(k)} \tag{31}$$

where $-P(k)$ and $-Q(k)$ are the arguments of the exponent in the k'th term of z_a and z_b, respectively.

We can compare this with the Orland and Mézard-Parisi results. In the limit $\mu_{1,2} \to \infty$ only large-k terms are relevant. Introducing $\xi = k e^{-\beta\mu_{1,2}}$ we can go to an integral representation. Adding and subtracting to z_a the expression $\sum_{k=1}^{\infty} \exp(-\xi)/k$ we regularize the small ξ behavior. We find that the μ contributions cancel, and obtain in the limit

$$\begin{aligned}\frac{\beta F}{N} = \sum_{p=1}^{\infty} \frac{Q_p P_p}{g_p} \frac{(-1)^{p-1}}{p} - \int_0^{\infty} \frac{d\xi}{\xi} [e^{-\xi} - \exp \sum_{p=1}^{\infty} \frac{P_p(-\xi)^p}{p!} e^{-\xi e^{-\beta A}}] \\ - \nu \int_0^{\infty} \frac{d\xi}{\xi} [e^{-\xi} - \exp \sum_{p=1}^{\infty} \frac{Q_p(-\xi)^p}{p!} e^{-\xi e^{-\beta B}}]\end{aligned} \tag{32}$$

In absence of bachelors ($A, B \to \infty$) and the limit of equal group sizes ($\nu = M/N \to 1$) this equation reduces to the result of Orland and Mezard-Parisi. Equation (32) was also derived by us using their method of taking derivatives rather than summing over residues.

The saddle point equations follow from eqs. (27), (28), and (29).

$$Q_p = \frac{p g_p}{p!} \int_0^{\infty} d\xi e^{-\xi e^{-\beta A}} \xi^{p-1} \exp \sum_{p=1}^{\infty} \frac{P_p(-\xi)^p}{p!} \tag{33}$$

and

$$P_p = \nu \frac{p g_p}{p!} \int_0^{\infty} d\xi e^{-\xi e^{-\beta B}} \xi^{p-1} \exp \sum_{p=1}^{\infty} \frac{Q_p(-\xi)^p}{p!} \tag{34}$$

IV. LOW TEMPERATURES

From eq. (15) one sees that $g_p = (p\hat{\beta})^{-(1+r)}$ is of order unity if we scale temperatures with N such that $\hat{T} \sim T N^{1/(1+r)}$ remains fixed. As discussed in section II, the free energy is not extensive but scales as $F = -T\log Z = -N^{-1/(r+1)}\hat{T}\log Z = N^{r/(r+1)}\hat{F}$ with intensive $\hat{F}$.

For having non-trivial behavior one also needs that A scales properly with N, $A = aN^{-1/(r+1)}$ so that $\beta A = \hat{\beta} a$, and similarly $\mu_{1,2} = \hat{\mu}_{1,2} N^{-1/(r+1)}$.

Inserting $\xi = \exp(\hat{\beta}\ell)$ we can now introduce the functions $\hat{P}(\ell) = P(\xi)$, etc. To simplify notation, we shall however, again denote $\hat{P}(\ell)$ by $P(\ell)$, etc. Thus

$$P(\ell) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1} P_p}{p!} e^{p\hat{\beta}\ell} \qquad Q(\ell) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1} Q_p}{p!} e^{p\hat{\beta}\ell} \tag{35}$$

At finite $\hat{T}$ they satisfy

$$P(\ell) = \nu \int_{-\infty}^{\infty} \mathrm{d}y B(\ell + y) e^{-Q(y)} e^{-e^{\hat{\beta}(y-a)}} \tag{36}$$

which for $\hat{T} \to 0$ reduces to

$$P(\ell) = \nu \int_{-\ell}^{a} \mathrm{d}y B(\ell + y) e^{-Q(y)} \tag{37}$$

Similarly

$$Q(\ell) = \int_{-\infty}^{\infty} \mathrm{d}y B(\ell + y) e^{-P(y)} e^{-e^{\hat{\beta}(y-b)}} \to \int_{-\ell}^{b} \mathrm{d}y B(\ell + y) e^{-P(y)} \tag{38}$$

with

$$\begin{aligned} B(\ell) &= \hat{T}^r \sum_{p=1}^{\infty} \frac{(-1)^{p-1} e^{p\hat{\beta}\ell}}{p^r p! p!} = \frac{\hat{\beta}}{\Gamma(r+1)} \int_0^{\infty} \mathrm{d}y y^r \sum_{p=1}^{\infty} \frac{(-1)^{p-1} e^{p\hat{\beta}(\ell - y)}}{p!(p-1)!} \\ &= \frac{\hat{\beta}}{\Gamma(r+1)} \int_0^{\infty} \mathrm{d}y y^r e^{\hat{\beta}(\ell-y)/2} J_1(2e^{\hat{\beta}(\ell-y)/2}) = \frac{1}{\Gamma(r+1)} \int_{-\infty}^{\hat{\beta}\ell} d\eta (\ell - \hat{T}\eta)^r e^{\eta/2} J_1(2e^{\eta/2}) \\ &\to \begin{cases} \frac{\ell^r}{\Gamma(r+1)} & \ell > 0 \\ 0 & \ell < 0 \end{cases} \end{aligned} \tag{39}$$

Here J_1 is a Bessel function.

A. Observables

The internal energy scales with N as

$$U = \frac{T}{\hat{T}} \hat{U} = \frac{1}{N} \left(\alpha^{r_a+1} (1-\alpha)^{r_b+1} \right)^{1/(r+1)} \hat{U} \tag{40}$$

The scaled internal energy $\hat{U} = \partial(\hat{\beta}\hat{F})/\partial\hat{\beta}$ consists of three terms

$$\hat{U} = \hat{U}_{ab} + \hat{U}_a + \hat{U}_b \tag{41}$$

The pair energy is

$$\begin{aligned} \frac{1}{N} \hat{U}_{ab} &= \frac{1+r}{\hat{\beta}} \sum_{p=1}^{\infty} \frac{Q_p P_p}{g_p} \frac{(-1)^{p-1}}{p} \\ &= (1+r) \int_{-\infty}^{\infty} d\ell \hat{P}(\ell) e^{-e^{\beta(\ell-a)}} e^{-\hat{P}(\ell)} \to (1+r) \int_{-\infty}^{a} d\ell \hat{P}(\ell) e^{-\hat{P}(\ell)} \\ &= (1+r)\nu \int_{-\infty}^{\infty} d\ell \hat{Q}(\ell) e^{-e^{\beta(\ell-b)}} e^{-\hat{Q}(\ell)} \to (1+r)\nu \int_{-\infty}^{b} d\ell \hat{Q}(\ell) e^{-\hat{Q}(\ell)} \end{aligned} \tag{42}$$

The male-bachelors energy is

$$\frac{1}{N} \hat{U}_a = a \int_{-\infty}^{\infty} d\ell \hat{\beta} e^{\hat{\beta}(\ell-c)} e^{-\hat{P}(\ell) - e^{\hat{\beta}(\ell-c)}} \to a e^{-\hat{P}(a)} \tag{43}$$

and the female-bachelors energy is

$$\frac{1}{N} \hat{U}_b = \nu b \int_{-\infty}^{\infty} d\ell \hat{\beta} e^{\hat{\beta}(\ell-b)} e^{-\hat{Q}(\ell) - e^{\hat{\beta}(\ell-b)}} \to \nu b e^{-\hat{Q}(b)} \tag{44}$$

The average fraction of married man is obtained as

$$\langle n\rangle = \frac{1}{N}\sum_{i=1}^{N}\langle n_i\rangle$$
$$= 1 - \int d\ell \beta e^{\beta\ell} e^{-e^{\beta(\ell-a)}} e^{-\hat{P}(\ell)} \to 1 - e^{-\hat{P}(a)} \tag{45}$$

and likewise the fraction of married woman

$$\langle m\rangle = \frac{1}{M}\sum_{j=1}^{M}\langle m_j\rangle \to 1 - e^{-\hat{Q}(b)} \tag{46}$$

The case of Orland, Mézard and Parisi is recovered for $\nu = 1$ and $a = b = \infty$ (infinite energy cost to remain single).

B. A special case at $T = 0$

Let us follow Orland and Mezard-Parisi and take $r = 0$. We assume equal costs $a = b$ for males and females to remain single. (This restriction is not necessary; our results can be generalized to the case $a \neq b$.) For this case eqs. (37,38) become

$$Q(\ell) = \int_{-\ell}^{a} \mathrm{d}y e^{-P(y)} \qquad P(\ell) = \nu \int_{-\ell}^{a} \mathrm{d}y e^{-Q(y)} \tag{47}$$

So that $P = Q = 0$ for $\ell \leq -a$. One gets

$$P'(\ell) = \nu e^{-Q(-\ell)} \qquad Q'(\ell) = e^{-P(-\ell)} \tag{48}$$

These equations can be combined in a single equation for P

$$P''(\ell) = P'(\ell) e^{-P(\ell)} \tag{49}$$

which can be integrated once. The solution reads

$$P(\ell) = \log\frac{(\gamma-1)e^{\gamma(\ell+a)}+1}{\gamma} \qquad Q(\ell) = \log[\frac{\nu}{\gamma}(\frac{e^{\gamma(\ell-a)}}{\gamma-1}+1)] \tag{50}$$

where γ is a parameter. It is obvious that $P(-a) = 0$, while $Q(-a) = 0$ requires that γ is the largest root of

$$e^{-2\gamma a} = (\gamma-1)(\frac{\gamma}{\nu}-1) \tag{51}$$

From this we derive

$$\langle n\rangle = \nu\langle m\rangle = 1 + \nu - \gamma \tag{52}$$

In the present case ($r = 0$) the energy $U \sim \mathcal{O}(N^0)$ is non-extensive. The contributions to the energy are at $T = 0$:

$$U_{ab} = \sum_{k=1}^{\infty}\left(\frac{\gamma^{1-k}}{k^2} + \gamma(1-\frac{\nu}{\gamma})^k[\frac{\ln(\gamma-\nu)}{k} - \frac{1}{k^2}]\right)$$
$$= \gamma\ln\frac{\nu}{\gamma}\ln(\gamma-\nu) + \gamma\sum_{k=1}^{\infty}\frac{1-(\gamma-\nu)^k}{k^2\gamma^k} \tag{53}$$

$$\frac{1}{N}\hat{U}_a = a(\gamma-\nu) \qquad \frac{1}{N}\hat{U}_b = a(\gamma-1) \tag{54}$$

We can now check some simple cases.

- For $a\to\infty$ and $\nu=1$ one has the Orland-Mézard-Parisi case with $\gamma=1$, so all men and women are coupled. If we choose $\alpha=1/2$, we recover the result $U_{ab}=\pi^2/12$, while there is no bachelors energy.

- If $a\gg 1$ at fixed $\nu>1$, it follows that $\gamma=\nu$, yielding and $\langle n\rangle=\nu\langle m\rangle=1$ and $U_a=0$, saying indeed that all N men are coupled to N out of the $M=\nu N$ women, a fraction $1/\nu$ of the total. There is no way to avoid the energy cost $U_b=a(\nu-1)=(M-N)A$ of the unpairable $M-N$ women, where A is the energy to pay for each bachelor. Likewise, for $\nu<1$ there are more men than women, and the result $\gamma=1$, $(1/\nu)\langle n\rangle=\langle m\rangle=1$ says that now all women are coupled to a fraction ν of the men. This time the surplus of males is responsible for the energy cost $(N-M)A$.

- For small a it holds that $U\sim U_{ab}\sim\langle n\rangle\sim a$. In the appendix we shall find that the fluctuations scale as $\delta U\sim a^{3/2}/\sqrt{N}$, implying $\delta U/U\sim\sqrt{a/N}$. As expected, it vanishes for $a\to 0$.

- In the limit $a\to 0$ it is energetically more advantageous to remain single. Though pairing may occur at finite T due to entropic reasons, the $T=0$ result $\gamma=1+\nu$ indeed implies $\langle n\rangle=\langle m\rangle=0$, $\hat U_{ab}=0$, expressing that no pairs are formed at $T=0$.

- Let us finally consider equally large groups ($\nu=1$) and finite scaled cost for remaining single, $a=b=\mathcal{O}(1)$. Then it follows that $\gamma=1+\exp(-\gamma a)$. The result $<n>=<m>=1-\exp(-\gamma a)$ expresses that a finite fraction of bachelors remains, because it is more advantageous that this fraction of the individuals is not paired. This aspect remains in the more general case $\nu\neq 1$ and $a\neq b$.

1. The entropy at low temperatures

A test for checking the correctness of the replica symmetric approach is to calculate the leading behavior of the entropy at low temperatures. For doing that, an expansion in powers of T is needed.

For simplicity we consider the case $r=0$, $a=b=\infty$, $\nu=1$. The kernel B from eq. (39) can be expanded in distribution sense

$$B(\ell)\equiv 1-J_0(2e^{\hat\beta\ell/2})=\theta(\ell)+a_1\hat T\delta(\ell)+a_2\hat T^2\delta'(\ell)+a_3\hat T^3\delta''(\ell)\cdots \tag{55}$$

Partial integrations reveal that

$$a_1=\hat\beta\int_{-\infty}^{\infty}\mathrm{d}\ell\,(B(\ell)-\theta(\ell))=-2\int_0^\infty \mathrm{d}x\,J_1(x)\log\frac{x}{2} \tag{56}$$

Similarly

$$a_2=-\hat\beta^2\int_{-\infty}^{\infty}\mathrm{d}\ell\,\ell\,(B(\ell)-\theta(\ell)-a_1\hat T\delta(\ell))=2\int_0^\infty \mathrm{d}x\,J_1(x)(\log\frac{x}{2})^2 \tag{57}$$

and

$$a_3=\frac{1}{2}\hat\beta^3\int_{-\infty}^{\infty}\mathrm{d}\ell\,\ell^2\,(B(\ell)-\theta(\ell)-a_1\hat T\delta(\ell)-a_2\delta'(\ell))=-\frac{4}{3}\int_0^\infty \mathrm{d}x\,J_1(x)(\log\frac{x}{2})^3 \tag{58}$$

It holds that [7]

$$\int_0^\infty \mathrm{d}x\left(\frac{x}{2}\right)^\mu J_1(x)=\frac{\Gamma(1+\frac12\mu)}{\Gamma(1-\frac12\mu)} \tag{59}$$

Expansion in powers of μ yields the needed integrals. One obtains

$$a_1=2\gamma_E;\qquad a_2=2\gamma_E^2;\qquad a_3=\frac{2}{3}\zeta(3)+\frac{4}{3}\gamma_E^3 \tag{60}$$

where $\gamma_E=0.577215$ is Euler's constant, and $\zeta(3)=\sum_{k=1}^\infty 1/k^3$ is a Rieman zeta function.

Eq. (36) reads for the present case

$$P(\ell) = \int_{-\infty}^{\infty} \mathrm{d}y B(\ell + y) e^{-P(y)} \tag{61}$$

Expanding

$$P(y) = P_0(y) + \hat{T} P_1(y) + \hat{T}^2 P_2(y) + \hat{T}^3 P_3(y) + \cdots \tag{62}$$

we find using (55)

$$\begin{aligned} P_1(\ell) &= -\int_{-\ell}^{\infty} \mathrm{d}y e^{-P_0(y)} P_1(y) + 2\gamma_E \frac{e^\ell}{e^\ell + 1} \\ P_2(\ell) &= \int_{-\ell}^{\infty} \mathrm{d}y e^{-P_0(y)} (-P_2(y) + \frac{1}{2} P_1(y)^2) - 2\gamma_E e^{-P_0(-\ell)} + 2\gamma_E^2 e^{-P_0(-\ell)} P_0'(-\ell) \end{aligned} \tag{63}$$

The solution reads

$$P_0(\ell) = \log(e^\ell + 1) \tag{64}$$

$$P_1(\ell) = \gamma_E \frac{e^\ell}{e^\ell + 1} \tag{65}$$

$$P_2(\ell) = \frac{1}{2} \gamma_E^2 \frac{e^\ell}{(e^\ell + 1)^2} \tag{66}$$

From eq. (32) we obtain the entropy

$$\begin{aligned} \frac{S}{2N} &= \hat{\beta} \int_{-\infty}^{\infty} \mathrm{d}\ell [e^{-e^{\hat{\beta}\ell}} - e^{-P(\ell)}] \\ &= \hat{\beta} \int_{-\infty}^{\infty} \mathrm{d}\ell \left([\theta(-\ell) - e^{-P_0(\ell)}] + [e^{-e^{\hat{\beta}\ell}} - \theta(-\ell) + \hat{T} e^{-P_0(\ell)} P_1(\ell)] + \hat{T}^2 [e^{-P_0(\ell)} (P_2(\ell) - \frac{1}{2} P_1(\ell)^2)] \right) \end{aligned} \tag{67}$$

The terms calculated so far lead to orders T^{-1}, T^0 and T. It turns out that all three prefactors vanish. Thus S is at least of order T^2. We have to go one step further. The equation for P_3 reads

$$\begin{aligned} P_3(\ell) = \int_{-\infty}^{\infty} \mathrm{d}y e^{-P_0(y)} \Big[&\theta(\ell + y) \{-P_3(y) + P_2(y) P_1(y) - \frac{1}{6} P_1^3(y)\} + 2\gamma_E \delta(\ell + y) \{-P_2(y) + \frac{1}{2} P_1^2(y)\} \\ &+ 2\gamma_E^2 \delta'(\ell + y) \{-P_1(y)\} + (\frac{2}{3}\zeta(3) + \frac{4}{3}\gamma_E^3) \delta''(\ell + y) \Big] \end{aligned} \tag{68}$$

Some tedious analysis reveals that the solution reads

$$P_3(\ell) = -\frac{\zeta(3)}{6} \frac{e^\ell}{e^\ell + 1} - \frac{\gamma_E^3}{6} \frac{e^\ell}{(e^\ell + 1)^2} + (\zeta(3) + \frac{1}{3}\gamma_E^3) \frac{e^\ell}{(e^\ell + 1)^3} \tag{69}$$

The next order contribution to the entropy reads

$$\frac{S}{2N} = \hat{T}^2 \int_{-\infty}^{\infty} \mathrm{d}y e^{-P_0(y)} \{P_3(y) - P_2(y) P_1(y) + \frac{1}{6} P_1^3(y)\} = -\hat{T}^2 P_3(\infty) \tag{70}$$

It follows that

$$S = \frac{1}{3} N \zeta(3) \hat{T}^2 \tag{71}$$

The specific heat therefore equals

$$C = \frac{2}{3} N \zeta(3) \hat{T}^2 \tag{72}$$

Notice that in S all contributions of order γ_E^3 have canceled. Therefore we could have simplified the calculation by neglecting γ_E. This makes it possible to extend the calculations to the case where a is finite.

C. Random costs for bachelors and $T = 0$

If the costs for bachelors are random with scaled densities $\rho_1(a)$ and $\rho_2(b)$, then eqs. (47) become

$$Q(\ell) = \int_{-\ell}^{\infty} \mathrm{d}y R_1(y) e^{-P(y)} \qquad P(\ell) = \nu \int_{-\ell}^{\infty} R_2(y) \mathrm{d}y e^{-Q(y)} \tag{73}$$

where

$$R_1(y) = \int_y^{\infty} \mathrm{d}a\rho_1(a); \qquad R_2(y) = \int_y^{\infty} \mathrm{d}a\rho_2(b) \tag{74}$$

This leads to the differential form

$$Q'(\ell) = R_1(-\ell)e^{-P(-\ell)}; \qquad P'(\ell) = \nu R_2(-\ell)e^{-Q(-\ell)} \tag{75}$$

and

$$P''(\ell) = -\frac{\rho_2(-\ell)}{R_2(-\ell)} P'(\ell) + R_1(\ell)e^{-P(\ell)}P'(\ell) \tag{76}$$

We have not been able to solve these equations for broad distributions ρ_1 and ρ_2. In case that the costs A_i and B_j for remaining bachelor take discrete values only, $\rho_1(a)$ and $\rho_2(b)$ are sums of delta-functions. Then the solution can be constructed by generalizing our previous solution in the segments between the δ-functions.

V. DISCUSSION

We have considered the bi-partite matching problem, where members of each group have a wish-list of non-negative real numbers ranking the possible partners from the other group. This is a polynomial (P) problem, for which fast algorithms are available. We have extended previous approaches [5] [6] to the case where groups have different size, members can remain single, and polygamy occurs. Our results can be extended to the case of random costs for remaining single or being polygamic, by averaging eq. (27) over these variables.

In section 2 we have given a scaling analysis for the problem. The interesting regime is where the ℓ's, the costs for pairings, are low. We assume that the probability density of the costs of males $\ell^{(a)}$, scales as a powerlaw, $(\ell^{(a)})^{r_a}$, and the one of females as $(\ell^{(b)})^{r_b}$. It is a standard result of statistics that the probability density for finding a couple with low cost $\ell = \ell^{(a)} + \ell^{(b)}$ then scales as ℓ^r with $r = r_a + r_b + 1$. The exponent relation $r + 1 = (r_a + 1) + (r_b + 1)$ expresses that both partners must have a low cost.

The typical cost can be estimated from $\ell_{typ}^{r+1} = 1/N$. Therefore the energy will scale as $U \sim N\ell_{typ} \sim N^{r/(r+1)}$. Bachelors and polygamists only play an interesting role when their costs ($A_i, B_j, \mu_{1,i}$ and $\mu_{2,j}$) scale as $U/N \sim N^{-1/(r+1)}$. If these costs have a larger magnitude, such options do not occur (provided the groups have equal size); if they are smaller, no pairs will be formed. Notice that these scalings of costs for bachelors and polygamists are not universal to the individual, but depend on the exponent r entering the distribution ℓ^r of costs for pairings in the whole society.

We have not worked out the role of polygamy at low temperatures. As it must play a role in case of unequal group sizes ($\nu \neq 1$) but absence of bachelors ($A = B = \infty$), we expect it generally to play a non-trivial role down to $T = 0$.

In a special case at $T = 0$ we generalized previously known exact solutions to include for bachelors and unequal group sizes. It could be checked that if the typical cost for being bachelor is of the right order of magnitude, quite a number of members of the groups will remain bachelor, even if the groups have equal sizes.

It is found that the entropy remains positive and vanishes quadratically in T. This result supports the validity of the replica symmetric approach.

A calculation of the variance of the free energy is performed in the Appendix. In principle replica symmetric approaches may lead to a negative (and thus wrong) prediction for this variance. The somewhat painful analysis shows that the replica symmetric Ansatz produces the correct sign. In the totally symmetric case of equal group sizes, equal weighting of males' and females' wishes, and in the absence of bachelors or polygamy (with parameters $\nu = 1$, $\alpha = 1/2$, $A = B = \mu_1 = \mu_2 = \infty$, $r_a = r_b = -1/2$) we derive $\overline{\delta U^2} = 0.442878/N$. It has the expected scaling in $1/N$. The correctness of the sign supports again the expectation that replica symmetry yields the correct result for this problem. In its turn replica symmetry is expected to lead at worst to a polynomial degeneracy of the ground state.

Our results can be summarized by saying that whether bachelors occur or not, depends not only on their intrinsic capacities, or lack thereof, but also on global aspects of pair formation in the society.

ACKNOWLEDGMENTS

The author's interest for this subject was raised in discussions with Yi-Cheng Zhang, which is gratefully acknowledged. The author is grateful for hospitality at the university of Fribourg, Switserland. He also thanks the ISI (Torino, Italy) and the ICTP (Trieste, Italy), where part of this work was done, for hospitality.

A. APPENDIX: VARIANCE OF THE FREE ENERGY

The present problem has only been considered within the replica symmetric Ansatz. To test the validity of this approach several checks are possible: a stability analysis, the sign of the entropy or the sign of the second cumulant of the free energy.

The physical free energy should have a positive variance. This implies that the leading term in N should be non-negative (if the leading term vanishes, then the fluctuations have typical amplitude less than $\sqrt{N}$). It was pointed out by Saakian and Nieuwenhuizen, that the $\mathcal{O}(n^2)$ of F_n can be used to check non-negativity of the variance [8]. Indeed, for $F = -T\log Z$ and $F_n = -T\log \overline{Z^n}$, it holds that

$$\beta^2\overline{(F-\overline{F})^2} = -2\beta\lim_{n\to 0}\frac{F_n - n\overline{F}}{n^2} \tag{77}$$

In their variational analysis of an interface in a disordered medium, Saakian and Nieuwenhuizen found that the variance is negative if no replica symmetry breaking is taken into account. For a finite number of breakings the variance becomes smaller in magnitude, but remains negative. Only for an infinity of breakings it becomes zero to leading order.

For simplicity we consider here the symmetric case $A = B$, $\nu = 1$, $\mu_1 = \mu_2 = \mu$. We first need the n^2 term of z_n^a, $z_a^{(2)}$, defined by eq. (25),

$$\begin{aligned} z_a^{(2)} &= \sum_{p=1}^{\infty}\frac{(-1)^p Q_p e^{-\beta\mu p}}{p}(\psi(p)-\psi(1)) + \sum_{k=1}^{\infty}\frac{e^{-Q(k)}(1-e^{-\beta(A+\mu_1)})^k}{k}(\psi(k)-\psi(1)) \\ &+ \sum_{k=1}^{\infty}\frac{e^{-Q(k)}(1-e^{-\beta(A+\mu_1)})^k}{k}\sum_{p=1}^{\infty}\frac{(-1)^{p-1}Q_p e^{-\beta\mu p}}{p} - \frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{e^{-Q(k)}(1-e^{-\beta(A+\mu_1)})^k}{k}\right)^2 \\ &- \sum_{k=1}^{\infty}\frac{e^{-Q(k)}(1-e^{-\beta(A+\mu_1)})^k}{k}\sum_{p=1}^{\infty}(-1)^{p-1}Q_p e^{-\beta\mu p}\frac{\Gamma(k+p)}{\Gamma(k)\,p!}(\psi(k+p)-\psi(k)) \end{aligned} \tag{78}$$

From eqs. (77), (20), (24), (23), (25), and (26) the variance follows as

$$\frac{\beta^2}{4N}\overline{\delta F^2} = -\frac{1}{2}\sum_{p=1}^{\infty}\frac{(-1)^p Q_p^2}{p g_p}(\psi(p)-\psi(1)) + z_a^{(2)} \tag{79}$$

In the first term we insert the equation of motion (30). We consider the obtained expression in the limit $\mu\to\infty$. As before, the only possible surviving terms are those where $e^{-\beta\mu}$ can pick up a factor k. The relevant terms are

$$\begin{aligned} \frac{\beta^2}{4N}\overline{\delta F^2} &= -\frac{1}{2}\sum_{k,p=1}^{\infty}\frac{(-1)^p Q_p e^{-p\beta\mu}\Gamma(k+p)}{\Gamma(k)p!}(\psi(p)-\psi(1))\frac{e^{-Q(k)}(1-e^{-\beta(A+\mu_1)})^k}{k} \\ &+ \sum_{k=1}^{\infty}\frac{e^{-Q(k)}(1-e^{-\beta(A+\mu_1)})^k}{k}(\psi(k)-\psi(1)) - \frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{e^{-Q(k)}(1-e^{-\beta(A+\mu_1)})^k}{k}\right)^2 \end{aligned} \tag{80}$$

In the limit $\mu\to\infty$ one obtains an integral representation involving the variable $\xi = ke^{-\beta\mu}$. We again add and subtract $e^{-\xi}$ to $e^{-Q(k)}(1-e^{-\beta(A+\mu_1)})^k$ to regularize the small ξ-behavior. This leads to

$$\begin{aligned} \frac{\beta^2}{4N}\overline{\delta F^2} &= -\frac{1}{2}\int_0^{\infty}\frac{\mathrm{d}\xi}{\xi}e^{-Q(\xi)}e^{-\xi e^{-\beta A}}\sum_{p=1}^{\infty}\frac{(-1)^p Q_p\xi^p}{p!}(\psi(p)-\psi(1)) + \int_0^{\infty}\frac{\mathrm{d}\xi}{\xi}(e^{-Q(\xi)}e^{-\xi e^{-\beta A}} - e^{-\xi})(\ln\xi+\beta\mu) \\ &+ (\frac{\beta^2\mu^2}{2} - \gamma_E\beta\mu) + \gamma_E\beta\mu - \frac{1}{2}\left(\beta\mu + \int_0^{\infty}\frac{\mathrm{d}\xi}{\xi}(e^{-Q(\xi)}e^{-\xi e^{-\beta A}} - e^{-\xi})\right)^2 \end{aligned} \tag{81}$$

where the last Euler's constant γ_E arises from $-\psi(1)$, while the first comes from an integral representation for the sum involving $\psi(k)$. Notice that all μ-dependent terms cancel, confirming that the situation without polygamy is well defined. Using (33) one has finally

$$\begin{aligned}\frac{\beta^2}{4N}\overline{\delta F^2} &= -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}\xi}{\xi} e^{-Q(\xi)} e^{-\xi e^{-\beta A}} \int_0^\infty \frac{\mathrm{d}\eta}{\eta} e^{-Q(\eta)} e^{-\eta e^{-\beta A}} B_2(\xi\eta) \\ &+ \int_0^\infty \frac{\mathrm{d}\xi}{\xi}(e^{-Q(\xi)}e^{-\xi e^{-\beta A}} - e^{-\xi})\ln\xi - \frac{1}{2}\left(\int_0^\infty \frac{\mathrm{d}\xi}{\xi}(e^{-Q(\xi)}e^{-\xi e^{-\beta A}} - e^{-\xi})\right)^2 \end{aligned} \tag{82}$$

where

$$B_2(\xi\eta) = \sum_{p=1}^\infty \frac{(-1)^p p g_p}{p!\,p!}(\xi\eta)^p(\psi(p) - \psi(1)) \tag{83}$$

As can be seen from the first non-vanishing contribution, the $p = 2$ term, B_2 is a positive function.

We now go to $T = 0$, so the internal energy equals the free energy. We calculate B_2 for the case $r = 0$. We have the scaling $T \sim \hat{T}/N$, $A = a/N$. We set $\xi = \exp\hat{\beta}\ell$, $\eta = \exp\hat{\beta}\ell'$ and insert in (83) the identity

$$\psi(p) - \psi(1) = -\oint \frac{\mathrm{d}z}{2\pi i z^p}\,\frac{\ln(1-z)}{1-z} \tag{84}$$

This brings us to a form where the p-sum can be carried out, yielding

$$B_2(\ell+\ell') = \hat{T}\oint \frac{\mathrm{d}z}{2\pi i}\,\frac{\ln(1-z)}{1-z}\left[1 - J_0\left(\frac{2e^{\hat{\beta}(\ell+\ell')/2}}{\sqrt{z}}\right)\right] \tag{85}$$

We can deform the z-contour and remain with the discontinuity along the real axis $z \geq 1$, which yields for $\ell + \ell' \geq 0$

$$\begin{aligned} B_2(\ell+\ell') &= \hat{T}\int_2^\infty \frac{\mathrm{d}z}{z-1}\left[1 - J_0\left(\frac{2e^{\hat{\beta}(\ell+\ell')/2}}{\sqrt{z}}\right)\right] + \mathcal{O}(\hat{T}) \\ &= \ell + \ell' + \mathcal{O}(\hat{T}) \end{aligned} \tag{86}$$

while $B_2 = 0$ for $\ell + \ell' \leq 0$.

We calculate the variance of the internal energy for the simplest, solvable case $r = 0$, $a = b$. To leading order, all terms in (82) are proportional to $\hat{\beta}^2 \sim \beta^2/N^2$, implying

$$\overline{\delta U^2} = \frac{2\alpha^{2r_a+2}(1-\alpha)^{2r_b+2}}{N}(-I_1 + 2I_2 - I_3^2) \tag{87}$$

Since $U \sim N^{r/(r+1)} = N^0$ it follows that $\delta U/U \sim N^{-1/2}$, as usual. It holds that

$$I_1 = \int_{-a}^{a}\mathrm{d}\ell\int_{-\ell}^{a}\mathrm{d}\ell' e^{-Q(\ell)-Q(\ell')}(\ell+\ell') \tag{88}$$

and

$$I_2 = \int_{-a}^{0}\mathrm{d}\ell\ell(e^{-Q(\ell)} - 1) + \int_0^a \mathrm{d}\ell\ell e^{-Q(\ell)} \tag{89}$$

and

$$I_3 = \int_{-a}^{0}\mathrm{d}\ell(e^{-Q(\ell)} - 1) + \int_0^a \mathrm{d}\ell e^{-Q(\ell)} = (\gamma - 1)a \tag{90}$$

For $a \to \infty$ (no bachelors) one has

$$I_1 = \int_{-\infty}^{\infty}\mathrm{d}\ell\ln(1+e^{\ell})\,\ln(1+e^{-\ell}) = 2*1.202056; \qquad I_2 = 2\int_0^\infty \ln(1+e^{-\ell}) = \frac{\pi^2}{6} = 1.644934; \qquad I_3 = 0 \tag{91}$$

Since $r=0$, boundedness of the entropy at low T implies that in this limit $I_3=0$, as can also be checked explicitly. Therefore the combination $-I_1+2I_2-I_3^2$ is positive. In the totally symmetric case (equal group sizes; equal role of males and females) with $\nu=1$ $\alpha=1/2$, $r_a=r_b=-1/2$ we thus find

$$\overline{\delta U^2}=\frac{0.442878}{N} \tag{92}$$

(Note that this implies for mono-parted matching problem with $r=0$, $p(\ell)=\exp(-\ell)$, considered by Mézard and Parisi [6]: $\overline{\delta U^2}=1.771512/N$). For $a\to 0$ it holds that $\gamma=1+e^{-\gamma a}\sim 2-2a$, and the prefactor

$$-I_1+2I_2-I_3^2\approx -\frac{4a^3}{3}+2(\frac{a^2}{2}-\frac{2}{3}a^3)-(a-2a^2)^2=\frac{4a^3}{3}+\mathcal{O}(a^4) \tag{93}$$

is still positive.

It is thus seen that the replica symmetric prediction for the (free) energy fluctuations has the correct sign. This supports the expectation that our replica symmetric solution is exact.

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